

Electronic Notes in Theoretical Computer Science 69 (2003)  
 URL: <http://www.elsevier.nl/locate/entcs/volume69.html> 22 pages

# Physical Traces: Quantum vs. Classical Information Processing

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## Abstract

Within the Geometry of Interaction (GoI) paradigm, we present a setting that enables qualitative differences between classical and quantum processes to be explored. The key construction is the physical interpretation/realization of the traced monoidal categories of finite dimensional vector spaces with tensor product as monoidal structure and of finite sets and relations with Cartesian product as monoidal structure, both of them providing a so-called wave-style GoI. The developments in this paper reveal that envisioning state update due to quantum measurement as a process provides a powerful tool for developing high-level approaches to quantum information processing.

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## 1 Introduction

Recall that a traced monoidal category is a symmetric monoidal category  $(\mathbf{C}, \otimes)$  such that for every morphism  $f : A \otimes C \rightarrow B \otimes C$  a trace  $Tr_{A,B}^C(f) : A \rightarrow B$  is specified and satisfies certain axioms [17]. We refer to the available literature for explicit definitions, e.g. [1,2,15,23]. These traced monoidal categories play an important role in Geometry of Interaction (GoI) and game

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<sup>3</sup> We thank Dusko Pavlovic, Phil Scott, Robert Seely, Peter Selinger and the two referees for inspiring remarks.

semantics, since every such traced monoidal category gives rise, via the GoI-construction, to a compact closed category in which composition corresponds to interaction of strategies [1].

Two qualitatively different families of GoI-style categorical semantics for concurrent processes are identified in [1], namely ‘particle-style’ and ‘wave-style’ models, based on the interpretation of the tensor as categorical coproduct and product respectively. The fundamentally different nature of these families of models is further exposed in the elaboration in [22] of Bainbridge’s [4] work on modelling flowcharts and networks, preceding the definition of traced monoidal categories by two decades.

The particle-style model is quite well understood and allows a “physical realization/interpretation” in terms of a particle traveling through a network (see below). The wave-style models require the interpretation of the trace as a fixpoint, and hence require a domain-theoretic context. They have a reasonable computational realization as dataflow networks, for which see [3]. Most mysterious are those examples in which the tensor is neither categorical product or sum. The most prominent examples are:

- The category of sets and relations with Cartesian product as the monoidal structure.
- The category of finite-dimensional vector spaces with the usual tensor product as the monoidal structure.

These are both compact-closed categories, but their status in either computational or physical terms is far from clear.

In this paper, we propose a physical realization for these categories, which we believe can be the basis for some interesting new directions in quantum computation. The main contributions of the present paper are as follows:

- We delineate qualitative differences between computational properties of classical and quantum systems, as incarnations of particle- and wave-style GoI-models respectively.
- We realize the multiplicative fragment of linear logic by means of quantum devices, thus providing some new insights into what is quantum about linear logic and what is linear about quantum logic.
- We develop a setting for quantum concurrency and quantum programming language development, implicitly aiming towards high-level approaches to quantum algorithm design.

Let us briefly sketch a particle-style model, namely the one that arises when considering the traced monoidal category  $(\mathbf{Rel}, +)$  of relations with disjoint union as tensor and with “feedback” as trace. In this category, for

$$R \subseteq (X + Z) \times (Y + Z),$$

the trace is formally given by

$$x \operatorname{Tr}_{X,Y}^Z(R) y \Leftrightarrow \exists z_1, \dots, z_n \in Z : x R z_1 R \dots R z_n R y.$$

The interpretation/realization of this category is as follows: We envision a particle traveling through a network where the objects  $X$  specify the states  $x \in X$  the particle “can have” at that stage (of traveling through the network) and the morphisms  $R \subseteq X \times Y$  are processes that impose a change of state from  $x$  to a state in  $\{y \in Y \mid xRy\}$ , which is possibly empty encoding that the process halts. The tensor  $X + Y$  is interpreted as disjoint union of state sets.  $R + R'$  encodes parallel composition where, depending on the initial state, either  $R$  or  $R'$  will act on the particle. The trace  $Tr_{X,Y}^Z(R)$  encodes feedback, that is, entering in a state  $x \in X$  the particle will either halt, exit at  $y \in Y$  or, exit at  $z_1 \in Z$  in which case it is fed back into  $R$  at the  $Z$  entrance, and so on, until it halts or exits at  $y \in Y$ .

It is easily seen that such a “one-dimensional” perspective cannot hold for the following traced monoidal categories:

- **(Rel,  $\times$ )** of relations with the Cartesian product as tensor and with as trace

$$x Tr_{X,Y}^Z(R)y \Leftrightarrow \exists z \in Z : (x, z)R(y, z),$$

for

$$R \subseteq (X \times Z) \times (Y \times Z).$$

- **(FDVec,  $\otimes$ )** of finite-dimensional vector spaces with the tensor product as tensor and with as trace

$$Tr_{\mathcal{V},\mathcal{W}}^{\mathcal{U}}(f) : \mathcal{V} \rightarrow \mathcal{W} : v_i \mapsto \sum_{k,\alpha} f_{i\alpha k\alpha} \cdot w_k$$

for

$$f : \mathcal{V} \otimes \mathcal{U} \rightarrow \mathcal{W} \otimes \mathcal{U} : v_i \otimes e_j \mapsto \sum_{k,l} f_{ijkl} \cdot (w_k \otimes e_l)$$

where  $\{v_i\}_i, \{w_k\}_k, \{e_l\}_l$  are the respective bases of  $\mathcal{V}, \mathcal{W}$  and  $\mathcal{U}$ .

Let us now recall the core of the GoI construction as outlined in [1]. Given a traced monoidal category  $(\mathbf{C}, \otimes, Tr)$  we define a new category  $\mathcal{G}(\mathbf{C})$  with objects given by pairs  $(A_+, A_-)$  of  $\mathbf{C}$ -objects. A  $\mathcal{G}(\mathbf{C})$ -morphism

$$f : (A_+, A_-) \rightarrow (B_+, B_-)$$

is a  $\mathbf{C}$ -morphism

$$\tilde{f} : A_+ \otimes B_- \rightarrow A_- \otimes B_+.$$

Given

$$f : (A_+, A_-) \rightarrow (B_+, B_-), \quad g : (B_+, B_-) \rightarrow (C_+, C_-),$$

their composition

$$h = (f; g) : (A_+, A_-) \rightarrow (C_+, C_-)$$

in  $\mathcal{G}(\mathbf{C})$  is given by the trace

$$\tilde{h} := Tr_{A_+ \otimes C_-, C_+ \otimes A_-}^{B_+ \otimes B_-}(\tilde{f} \otimes \tilde{g}).$$

Via this construction one obtains a compact closed category, that is, a \*-autonomous category with self-dual tensor, where a \*-autonomous category is

a symmetric monoidal closed category with a dualizing object. See for example [15] for an overview on these. Another crucial property of this construction is that the category  $\mathbf{C}$  embeds fully, faithfully and preserving the trace into  $\mathcal{G}(\mathbf{C})$ , identifying  $A$  with  $(A, I)$  where  $I$  is the unit for the tensor. We thus embed the category  $\mathbf{C}$  with “sequential application” as composition into the compact closed category  $\mathcal{G}(\mathbf{C})$  with “parallel interaction” as composition.

The essence of the present paper lies in the construction of an interaction category with true physical processes as morphisms which compose in a concurrent fashion. Thus, even though  $\mathbf{FDVec}$  is itself already compact closed, it makes sense to construct the “larger” compact closed category  $\mathcal{G}(\mathbf{FDVec})$ . Moreover, the  $*$ -operation of the compact closed structure of  $\mathbf{FDVec}$  has no direct physical implementation, but will turn out to encode an instance of the *duality between what measures and what is measured*. The construction for  $\mathbf{FDVec}$  induces a realization/interpretation of the category  $(\mathbf{FRel}, \times)$  of finite sets and relations with the Cartesian product as monoidal structure. This will enable us to make a qualitative comparison between quantum and classical process networks via the “descent”

$$\text{quantum}(\mathbf{FDVec}, \otimes) \mapsto \text{pseudo-quantum}(\mathbf{Rel}, \times) \mapsto \text{classical}(\mathbf{Rel}, +).$$

From a mathematical perspective, part of the formal investigation consists of “how much can be done when only using projectors on a Hilbert space”, which physically translates as “how much can be done in terms of state update due to measurements performed on quantum systems”. Although considering state update as a truly dynamical process introduces an uncertainty on the actual realization of these processes, it allows a new spectrum of applications since projections are *not* isomorphisms of the Hilbert space projection lattice — by contrast to unitary transformations which are isomorphisms of that structure. Furthermore, in this paper we also prove that any linear map on a Hilbert space, including of course unitary transformations, can be realized in terms of projections.

The whole development reveals that projections, or in physical terms, quantum state update due to measurement, provide a powerful tool for high-level approaches to quantum information processing. In this context, we also mention [8]. Thus, in comparison with the current paradigm with respect to quantum computation, involving a setup consisting of a preparation, a unitary operation and a measurement, we take a radically different perspective. In view of the fact that most of the power of quantum computation with respect to the known algorithms exploits quantum entanglement, we also mention that in our setting primitive operations are “specifications of entanglement”, subject to a Linear Logical type system.

The primary technical contributions of this paper are presented in Sections 2.2 and 2.3. In Section 2.1 we recall some basics on vector spaces. Since in the finite-dimensional case it is harmless to assume that each complex vector space is in fact a Hilbert space, they provide the appropriate setting for elementary

superselection free quantum theory, to which we provide a brief introduction in Section 3.1. The primary conceptual results are presented in Section 3.2 and 3.3. In Section 3.4 and 4 we discuss possible applications of our results.

## 2 Constructions for finite-dimensional vector spaces

First we recall some basics on finite-dimensional vector spaces and establish some notation that we will need in this paper. Then we derive the two key lemmas for this paper.

### 2.1 Finite-dimensional vector spaces and projectors

Recall that a finite-dimensional Hilbert space is a complex vector space  $(\mathcal{H}, \mathbf{0}, \cdot, +, \mathbb{C})$  equipped with an inner product  $\langle - | - \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$  that satisfies

$$\begin{aligned} \langle w | c_1 \cdot v_1 + c_2 \cdot v_2 \rangle &= c_1 \langle w | v_1 \rangle + c_2 \langle w | v_2 \rangle, \\ \langle c_1 \cdot w_1 + c_2 \cdot w_2 | v \rangle &= \bar{c}_1 \langle w_1 | v \rangle + \bar{c}_2 \langle w_2 | v \rangle, \\ \overline{\langle w | v \rangle} &= \langle v | w \rangle, \quad \langle w | w \rangle = 0 \Rightarrow w = \mathbf{0} \quad \text{and} \quad \langle w | w \rangle \geq 0, \end{aligned}$$

where the latter allows us to define a norm on  $\mathcal{H}$  as  $| - | := \sqrt{\langle - | - \rangle}$ . We introduce an orthogonality relation  $\perp \subseteq \mathcal{V} \times \mathcal{V}$  such that  $v \perp w \Leftrightarrow \langle v | w \rangle = 0$  and given a subspace  $V$  of  $\mathcal{V}$  its orthocomplement is

$$V^\perp := \{w \in \mathcal{V} \mid \forall v \in V : v \perp w\}.$$

Every finite-dimensional complex vector space extends to a Hilbert space via choice of an inner product. An orthonormal base is a set of vectors  $\{v_i\}_i$  such that  $\langle v_i | v_j \rangle = \delta_{ij}$  where  $\delta_{ij}$  is the Kronecker delta, that is,

$$\delta_{ij} = 1 \quad \text{when} \quad i = j \quad \text{and} \quad \delta_{ij} = 0 \quad \text{when} \quad i \neq j.$$

A *projector* is an idempotent self-adjoint linear endomap  $P : \mathcal{V} \rightarrow \mathcal{V}$  on a Hilbert space  $\mathcal{V}$ , that is respectively, for  $v, w \in \mathcal{V}$ ,

$$P(P(v)) = P(v) \quad \text{and} \quad \langle P(v) | w \rangle = \langle v | P(w) \rangle.$$

As an example, given a unit vector  $v \in \mathcal{V}$ , that is  $|v| = 1$ , the map

$$P_v : \mathcal{V} \rightarrow \mathcal{V} : w \mapsto \langle v | w \rangle \cdot v$$

defines a projector. The fixpoints of  $P_v$  constitute the one-dimensional subspace spanned by  $v$ .

The functionals  $\mathcal{V}^*$ , i.e. the linear maps  $f : \mathcal{V} \rightarrow \mathbb{C}$ , constitute a vector space isomorphic to  $\mathcal{V}$ . However, there is in general no canonical isomorphism that connects them. Indeed, since we have  $\langle c \cdot v | - \rangle = \bar{c} \langle v | - \rangle$  the canonical

correspondence is anti-linear instead of linear. Thus, given a base  $\{v_i\}_i$  of  $\mathcal{V}$ , specification of an isomorphism as  $\mathcal{V} \rightarrow \mathcal{V}^* : v_i \mapsto \langle v_i | - \rangle$  depends on the choice of the base. A functional  $\langle v | - \rangle : \mathcal{V} \rightarrow \mathbb{C}$  defines a projector  $\langle \frac{v}{|v|} | - \rangle \cdot \frac{v}{|v|} : \mathcal{V} \rightarrow \mathcal{V}$  via composition with the injection  $\mathbb{C} \rightarrow \mathcal{V} : c \mapsto \frac{c}{|v|^2} \cdot v$ .<sup>4</sup> For a linear map  $f : \mathcal{V} \rightarrow \mathcal{W}$  and orthonormal bases  $\{v_i\}_i$  and  $\{w_j\}_j$  we have

$$f = \sum_i f_{ij} \langle v_i | - \rangle \cdot w_j \quad \text{given that} \quad f(v_i) = f_{ij} \cdot w_j.$$

Any pair of complex vector spaces  $\mathcal{V}$  and  $\mathcal{W}$  admits a tensor product, that is, a pair consisting of a vector space  $\mathcal{V} \otimes \mathcal{W}$  and a bilinear map  $h : \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{V} \otimes \mathcal{W}$  such that for any other bilinear map  $f : \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{U}$  there exists a unique  $g : \mathcal{V} \otimes \mathcal{W} \rightarrow \mathcal{U}$  with  $f = h; g$ . This tensor product equips the category of finite-dimensional complex vector spaces and linear maps with a monoidal structure with  $\mathbb{C}^{(1)}$  as unit, since given

$$h : \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{V} \otimes \mathcal{W} \quad \text{and} \quad h' : \mathcal{V}' \times \mathcal{W}' \rightarrow \mathcal{V}' \otimes \mathcal{W}',$$

and two linear maps  $f : \mathcal{V} \rightarrow \mathcal{V}'$  and  $g : \mathcal{W} \rightarrow \mathcal{W}'$ , their tensor product

$$f \otimes g : \mathcal{V} \otimes \mathcal{W} \rightarrow \mathcal{V}' \otimes \mathcal{W}'$$

is uniquely defined due to universality of  $h$  with respect to  $(f \times g); h'$ .

We can construct a tensor product for vector spaces  $\mathcal{V}$  and  $\mathcal{W}$  as  $h : (v_i, w_j) \mapsto v_i \otimes w_j$  with  $\mathcal{V} \otimes \mathcal{W}$  being the vector space spanned by  $\{v_i \otimes w_j\}_{i,j}$ . Identifying

$$\left( \sum_i c_i \cdot v_i \right) \otimes \left( \sum_i d_i \cdot w_i \right) \quad \text{and} \quad \sum_{i,j} c_i d_j \cdot (v_i \otimes w_j),$$

this construction does not depend on the choice of orthonormal base, in particular, for arbitrary  $v \in \mathcal{V}$  and  $w \in \mathcal{W}$  we have  $h(v, w) = v \otimes w$ . We can define an inner product on  $\mathcal{V} \otimes \mathcal{W}$  via

$$\langle v \otimes v' | w \otimes w' \rangle := \langle v | w \rangle \langle v' | w' \rangle,$$

so

$$\left\langle \sum_{i,j} c_{ij} \cdot (v_i \otimes w_j) \mid \sum_{k,l} c_{kl} \cdot (v'_k \otimes w'_l) \right\rangle = \sum_{i,j,k,l} \bar{c}_{ij} c_{kl} \langle v_i | v'_k \rangle \langle w_j | w'_l \rangle.$$

<sup>4</sup> The correspondence between projectors on one-dimensional subspaces of a finite-dimensional complex vector space and the vector space itself is actually rather one between the projective space of subspaces, since its points are exactly the one-dimensional subspaces of the vector space. More generally, the (complete) lattice of all subspaces of the finite-dimensional complex vector space is isomorphic to that of its projectors ordered by  $P \leq Q \Leftrightarrow P; Q = P$ . This matter becomes crucial in the passage to quantum theory where states correspond with one-dimensional subspaces.

When both  $\{v_i\}_i$  and  $\{w_j\}_j$  are orthonormal bases then  $\{v_i \otimes w_j\}_{i,j}$  is again orthonormal with respect to this inner product.

The general form of elements of  $\mathcal{V}^* \otimes \mathcal{W}$  and  $\text{Hom}(\mathcal{V}, \mathcal{W})$  respectively is

$$\sum_{i,j} c_{ij} \cdot (\langle v_i | - \rangle \otimes w_j) \quad \text{and} \quad \sum_i c_{ij} \langle v_i | - \rangle \cdot w_j$$

and thus we obtain an isomorphism of vector spaces when providing the set  $\text{Hom}(\mathcal{V}, \mathcal{W})$  with its canonical vector space structure. Note also that we have  $\mathcal{V}^* \otimes \mathcal{W}^* \cong (\mathcal{V} \otimes \mathcal{W})^*$  via identification of  $f_i \otimes g_j$ , with  $\{f_i : \mathcal{V} \rightarrow \mathbb{C}\}_i$  and  $\{g_j : \mathcal{W} \rightarrow \mathbb{C}\}_j$  respective bases for  $\mathcal{V}^*$  and  $\mathcal{W}^*$ , and the unique functional  $f_i * g_j : \mathcal{V} \otimes \mathcal{W} \rightarrow \mathbb{C}$  that arises due to universality of  $h$  within

$$\begin{array}{ccc} \mathcal{V} \otimes \mathcal{W} & \xrightarrow{f_i * g_j} & \mathbb{C} \\ \uparrow h & \nearrow f_i g_j & \\ \mathcal{V} \times \mathcal{W} & & \end{array}$$

where

$$f_i g_j : \mathcal{V} \times \mathcal{W} \rightarrow \mathbb{C} : (v, w) \mapsto f_i(v) g_j(w).$$

The definition for an inner product on  $\mathcal{V} \otimes \mathcal{W}$  then embodies this fact as  $\langle v \otimes w | - \otimes - \rangle := \langle v | - \rangle \langle w | - \rangle$  when expressing functionals in terms of the inner product.

Categorically, the correspondence between linear maps and the tensor product arises due to the fact that the category of finite-dimensional complex vector spaces and linear maps **FDVec** is compact closed plus the observation that the internal homs of the form  $[\mathcal{V}, \mathbb{C}]$  exactly define the vector space of functionals, since the dualizing object of the  $*$ -autonomous structure and the monoidal unit  $\mathbb{C}$  coincide, i.e.  $\mathbb{C} \cong \mathbb{C}^*$ . Thus,

$$[\mathcal{V}, \mathcal{W}] = [\mathcal{V}, [\mathcal{W}^*, \mathbb{C}]] \cong [\mathcal{V} \otimes \mathcal{W}^*, \mathbb{C}] = (\mathcal{V} \otimes \mathcal{W}^*)^* \cong \mathcal{V}^* \otimes \mathcal{W},$$

the last isomorphism being compact closedness for a  $*$ -autonomous category.

## 2.2 Implementing the **FDVec** trace via projectors

Let  $\{e_i\}_i$  be a base of a vector space  $\mathcal{U}$  and let  $\{\bar{e}_i\}_i$  be the corresponding linear functionals in  $\mathcal{U}^*$  via anti-linear correspondence  $e_i \mapsto \bar{e}_i$ . More generally, denote by  $\bar{u}$  the vector  $\langle u | - \rangle \in \mathcal{U}^*$  that corresponds with  $u \in \mathcal{U}$ . Let

$$P_{\mathcal{U}^*} := P_{\frac{1}{\sqrt{N}}, \sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha}} : \mathcal{U} \otimes \mathcal{U}^* \rightarrow \mathcal{U} \otimes \mathcal{U}^*,$$

that is

$$P_{\mathcal{U}^*}(v) = \frac{1}{N} \left\langle \sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha} \mid v \right\rangle \cdot \sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha},$$

where  $N$  denotes the dimension of  $\mathcal{U}$  so that  $|\frac{1}{\sqrt{N}} \cdot \sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha}| = 1$ .

Considering the base vectors  $\{\bar{e}_i\}_i$  in  $\mathcal{U}^*$  at the right side of the tensor rather than the base  $\{e_i\}_i$  makes the vector  $\sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha}$  base independent, that is, this vector has the same coordinates with respect to any base of the form  $\{e'_i \otimes \bar{e}'_j\}_{i,j}$  of  $\mathcal{U} \otimes \mathcal{U}^*$ . This motivates the notation  $P_{\mathcal{U}^*}$ . (the  $*$  will become clear further)

We can realize the **FDVec** traces by means of projectors as follows.

**Lemma 1.** *For any  $f : \mathcal{V} \otimes \mathcal{U} \rightarrow \mathcal{W} \otimes \mathcal{U}$  and  $v \in \mathcal{V}$  we have for the map*

$$\tau = N \cdot (f \otimes id_{\mathcal{U}^*} ; id_{\mathcal{W}} \otimes P_{\mathcal{U}^*}) : \mathcal{V} \otimes \mathcal{U} \otimes \mathcal{U}^* \rightarrow \mathcal{W} \otimes \mathcal{U} \otimes \mathcal{U}^*$$

that

$$(1) \quad \tau \left( v \otimes \left( \sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha} \right) \right) = Tr_{\mathcal{V}, \mathcal{W}}^{\mathcal{U}}(f)(v) \otimes \left( \sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha} \right).$$

Equivalently, setting

$$\theta = (id_{\mathcal{W}} \otimes P_{\mathcal{U}^*} ; \tau) : \mathcal{V} \otimes \mathcal{U} \otimes \mathcal{U}^* \rightarrow \mathcal{W} \otimes \mathcal{U} \otimes \mathcal{U}^*$$

we have for  $u \in \mathcal{U} \otimes \mathcal{U}^*$  that

$$(2) \quad \theta(v \otimes u) = Tr_{\mathcal{V}, \mathcal{W}}^{\mathcal{U}}(f)(v) \otimes P_{\mathcal{U}^*}(u).$$

**Proof.** Since we have

$$(f \otimes id_{\mathcal{U}^*}) \left( v_i \otimes \left( \sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha} \right) \right) = \sum_{k,l,\alpha} f_{i\alpha kl} \cdot (w_k \otimes e_l \otimes \bar{e}_{\alpha})$$

it follows that

$$\begin{aligned} & \tau \left( v_i \otimes \sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha} \right) \\ &= N \cdot \sum_{k,l,\alpha} f_{i\alpha kl} \cdot \left( w_k \otimes P_{\mathcal{U}^*}(e_l \otimes \bar{e}_{\alpha}) \right) \\ &= \sum_{k,l,\alpha} f_{i\alpha kl} \cdot \left( w_k \otimes \left\langle \sum_{\gamma} e_{\gamma} \otimes \bar{e}_{\gamma} \mid e_l \otimes \bar{e}_{\alpha} \right\rangle \cdot \left( \sum_{\beta} e_{\beta} \otimes \bar{e}_{\beta} \right) \right) \\ &= \sum_{k,l,\alpha} f_{i\alpha kl} \cdot \left( w_k \otimes \left( \sum_{\gamma} \delta_{\gamma l} \delta_{\gamma \alpha} \right) \cdot \left( \sum_{\beta} e_{\beta} \otimes \bar{e}_{\beta} \right) \right) \\ &= \left( \sum_{k,l,\alpha} \delta_{\alpha l} f_{i\alpha kl} \cdot w_k \right) \otimes \left( \sum_{\beta} e_{\beta} \otimes \bar{e}_{\beta} \right) \\ &= Tr_{\mathcal{V}, \mathcal{W}}^{\mathcal{U}}(f)(v_i) \otimes \left( \sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha} \right). \end{aligned}$$

By linearity eq.(1) then follows. We obtain the same result by considering the unit vector  $\frac{1}{\sqrt{N}} \cdot \sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha}$  in stead of  $\sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha}$  in  $v_i \otimes (\sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha})$ , or more generally, by considering any element of the subspace spanned by the



unit vector  $\frac{1}{\sqrt{N}} \cdot \sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha}$ , that is, any element in the image of  $P_{\mathcal{U}^*}$ . So for arbitrary  $u \in \mathcal{U} \otimes \mathcal{U}^*$  eq.(2) follows.  $\square$

Thus the map

$$\tau\left(- \otimes \left(\sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha}\right)\right) : \mathcal{V} \rightarrow \mathcal{W} \otimes \mathcal{U} \otimes \mathcal{U}^*$$

produces  $Tr_{\mathcal{V}, \mathcal{W}}^{\mathcal{U}}(f)(-)$  as the first component of a pure tensor with  $\sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha}$  as *context* (= the remaining component). Since we have

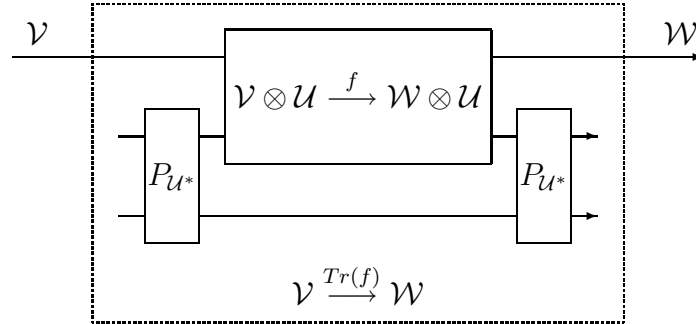
$$v \otimes w = \mathbf{0} \Leftrightarrow v = \mathbf{0} \text{ or } w = \mathbf{0}$$

the trace is also produced by the map

$$\theta(- \otimes u) : \mathcal{V} \rightarrow \mathcal{W} \otimes \mathcal{U} \otimes \mathcal{U}^*$$

as the first component of a pure tensor with  $P_{\mathcal{U}^*}(u)$  as context provided  $P_{\mathcal{U}^*}(u) \neq \mathbf{0}$ , that is,  $u \not\perp \sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha}$  (including the case  $u = \mathbf{0}$ ). The function  $Tr_{\mathcal{V}, \mathcal{W}}^{\mathcal{U}}(f)(-)$  is encoded up to rescaling and a phase factor since in general  $(c \cdot v) \otimes w = v \otimes (c \cdot w)$ .

Abstracting over the dimension  $N$  we can represent eq.(2) graphically as



where the arrows that start and end within the dotted lines embody the above mentioned contexts.

Incorporating the dimension  $N$  in order to obtain projections seems an unnecessary complication and indeed, from a purely mathematical perspective it is. However, expressing things in terms of true projectors will be crucial to us when considering a physical realisation of traces of vector spaces in the next section.

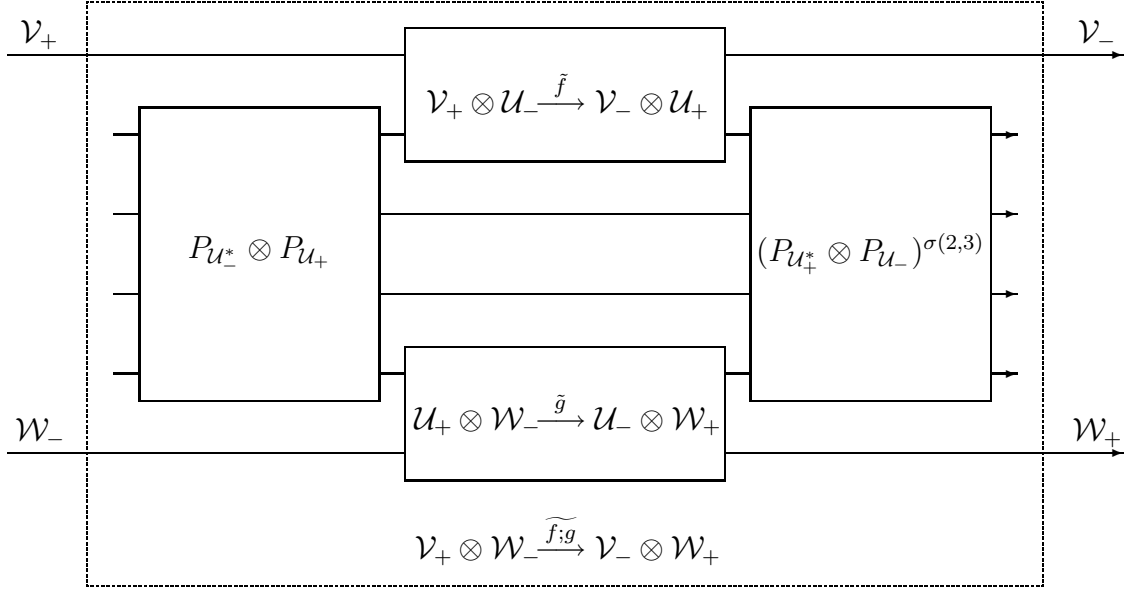
The fact that we have a traced monoidal category assures embedding in a compact closed category via the geometry of interaction construction. One verifies that composition via interaction for  $\tilde{f} : \mathcal{V}_+ \otimes \mathcal{U}_- \rightarrow \mathcal{V}_- \otimes \mathcal{U}_+$  and  $\tilde{g} : \mathcal{U}_+ \otimes \mathcal{W}_- \rightarrow \mathcal{U}_- \otimes \mathcal{W}_+$  has the following interpretation/realization (via replacing the trace by its above explicit interpretation/realization)

$$id_{\mathcal{V}_+} \otimes P_{\mathcal{U}_-}^* \otimes P_{\mathcal{U}_+} \otimes id_{\mathcal{W}_-} ; \tilde{f} \otimes id_{\mathcal{U}_-}^* \otimes id_{\mathcal{U}_+} \otimes \tilde{g} ; id_{\mathcal{V}_-} \otimes (P_{\mathcal{U}_+}^* \otimes P_{\mathcal{U}_-})^{\sigma(2,3)} \otimes id_{\mathcal{W}_+}$$

where by omitting  $*$  in  $P_{\mathcal{U}_+}$  and  $P_{\mathcal{U}_-}$  we refer to the fact that the two components in the tensor are swapped, that is

$$P_{\mathcal{U}} := P_{\frac{1}{\sqrt{N}} \cdot \sum_{\alpha} \bar{e}_{\alpha} \otimes e_{\alpha}} : \mathcal{U}^* \otimes \mathcal{U} \rightarrow \mathcal{U}^* \otimes \mathcal{U},$$

and by  $\sigma(2, 3)$  we mean that the second and the third component in the tensor are swapped. We then envision this operator as acting on  $v_+ \otimes u \otimes w_-$  where  $u \in \mathcal{U}_- \otimes \mathcal{U}_-^* \otimes \mathcal{U}_+^* \otimes \mathcal{U}_+$ . Graphically this represents as



In view of **FDVec** being compact closed, that is, its trace can be defined as

$$\mathcal{V} \xrightarrow{id_{\mathcal{V}} \otimes \eta_{\mathcal{U}}} \mathcal{V} \otimes \mathcal{U} \otimes \mathcal{U}^* \xrightarrow{f \otimes id_{\mathcal{U}^*}} \mathcal{W} \otimes \mathcal{U} \otimes \mathcal{U}^* \xrightarrow{id_{\mathcal{W}} \otimes \sigma} \mathcal{W} \otimes \mathcal{U}^* \otimes \mathcal{U} \xrightarrow{id_{\mathcal{W}} \otimes \epsilon_{\mathcal{U}}} \mathcal{W}$$

where corresponding unit and counit are

$$\eta_{\mathcal{U}} : \mathbb{C} \rightarrow \mathcal{U} \otimes \mathcal{U}^* : 1 \mapsto \sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha} \quad \epsilon_{\mathcal{U}} : \mathcal{U} \otimes \mathcal{U}^* \rightarrow \mathbb{C} : (v, \bar{w}) \mapsto \langle w | v \rangle,$$

the crucial part of this construction boils down to the fact that the counit can be interpreted as

$$\epsilon_{\mathcal{U}}(e_i \otimes \bar{e}_j) = \langle e_j | e_i \rangle = \delta_{ij} = \sum_{\alpha} \delta_{\alpha i} \delta_{\alpha j} = \left\langle \sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha} \mid e_i \otimes \bar{e}_j \right\rangle$$

or equivalently,

$$\epsilon_{\mathcal{U}} \otimes \sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha} = \left\langle \sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha} \mid - \right\rangle \cdot \sum_{\alpha} e_{\alpha} \otimes \bar{e}_{\alpha} = N \cdot P_{\mathcal{U}^*}.$$

### 2.3 Implementing linear functions via projections

Given a linear map  $f : \mathcal{V} \rightarrow \mathcal{W}$  and orthonormal bases  $\{v_i\}_i$  and  $\{w_j\}_j$  set

$$P_f := P_{\frac{1}{\sqrt{M}} \cdot \sum_{i,j} f_{ij} \cdot (\bar{v}_i \otimes w_j)} : \mathcal{V}^* \otimes \mathcal{W} \rightarrow \mathcal{V}^* \otimes \mathcal{W},$$

that is

$$P_f(v) = \frac{1}{M} \left\langle \sum_{i,j} f_{ij} \cdot (\bar{v}_i \otimes w_j) \mid v \right\rangle \cdot \sum_{i,j} f_{ij} \cdot (\bar{v}_i \otimes w_j),$$

where  $M = |\sum_{i,j} f_{ij} \cdot (\bar{v}_i \otimes w_j)|^2$ .

The notation  $P_f$  is justified by base independence of  $\frac{1}{\sqrt{M}} \cdot \sum_{i,j} f_{ij} \cdot (\bar{v}_i \otimes w_j)$  due to the canonical correspondence between  $\mathcal{V}^* \otimes \mathcal{W}$  and  $[\mathcal{V}, \mathcal{W}]$ . In fact, the projector  $P_{\mathcal{U}}$  embodies a particular case of this for  $f := id_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{U}$  (and  $P_{\mathcal{U}^*}$  for  $f := id_{\mathcal{U}^*} : \mathcal{U}^* \rightarrow \mathcal{U}^*$ ) given that

$$id_{\mathcal{U}} = \sum_{i,j} \delta_{ij} \cdot (\langle e_i | - \rangle \cdot e_j) \quad \text{and} \quad N = |\sum_{i,j} \delta_{ij} \cdot (\bar{e}_i \otimes e_j)|^2$$

where obviously

$$\sum_{i,j} \delta_{ij} \cdot (\bar{e}_i \otimes e_j) = \sum_{\alpha} \bar{e}_{\alpha} \otimes e_{\alpha}.$$

It is our aim to produce arbitrary linear maps using only projections.

**Lemma 2.** *For any  $f : \mathcal{V} \rightarrow \mathcal{W}$  and  $v \in \mathcal{V}$  we have for the map*

$$\xi = N \cdot (P_{\mathcal{V}^*} \otimes id_{\mathcal{W}}) : \mathcal{V} \otimes \mathcal{V}^* \otimes \mathcal{W} \rightarrow \mathcal{V} \otimes \mathcal{V}^* \otimes \mathcal{W}$$

that

$$(3) \quad \xi \left( v \otimes \left( \sum_{i,j} f_{ij} \cdot (\bar{v}_i \otimes w_j) \right) \right) = \left( \sum_{\alpha} v_{\alpha} \otimes \bar{v}_{\alpha} \right) \otimes f(v).$$

Equivalently, setting

$$\zeta = (id_{\mathcal{V}} \otimes P_f ; \xi) : \mathcal{V} \otimes \mathcal{V}^* \otimes \mathcal{W} \rightarrow \mathcal{V} \otimes \mathcal{V}^* \otimes \mathcal{W}$$

we have for  $u \in \mathcal{V}^* \otimes \mathcal{W}$

$$(4) \quad \zeta(v \otimes u) = K \cdot \left( \sum_{\alpha} v_{\alpha} \otimes \bar{v}_{\alpha} \right) \otimes f(v)$$

where  $K \in \mathbb{C}$  only depends on  $u$  and not on  $v$ .

**Proof.** We have

$$\begin{aligned} & \xi \left( v_k \otimes \left( \sum_{i,j} f_{ij} \cdot (\bar{v}_i \otimes w_j) \right) \right) \\ &= N \cdot \sum_{i,j} P_{\mathcal{V}^*}(v_k \otimes f_{ij} \cdot \bar{v}_i) \otimes w_j \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i,j,\gamma} \left( \langle v_\gamma \otimes \bar{v}_\gamma | v_k \otimes f_{ij} \cdot \bar{v}_i \rangle \cdot \left( \sum_{\alpha} v_\alpha \otimes \bar{v}_\alpha \otimes w_j \right) \right) \\
 &= \sum_{i,j,\gamma} f_{ij} \delta_{\gamma k} \delta_{\gamma i} \cdot \left( \sum_{\alpha} v_\alpha \otimes \bar{v}_\alpha \otimes w_j \right) \\
 &= \sum_j f_{kj} \cdot \left( \sum_{\alpha} v_\alpha \otimes \bar{v}_\alpha \otimes w_j \right) \\
 &= \left( \sum_{\alpha} v_\alpha \otimes \bar{v}_\alpha \right) \otimes \sum_j f_{kj} \cdot w_j.
 \end{aligned}$$

By linearity eq.(3) then follows. Since moreover

$$\begin{aligned}
 P_f(u) &= \frac{1}{M} \left\langle \sum_{i,j} f_{ij} \cdot (\bar{v}_i \otimes w_j) \mid u \right\rangle \cdot \sum_{i,j} f_{ij} \cdot (\bar{v}_i \otimes w_j) \\
 &= K \cdot \sum_{i,j} f_{ij} \cdot (\bar{v}_i \otimes w_j)
 \end{aligned}$$

for

$$K = \frac{\left\langle \sum_{i,j} f_{ij} \cdot (v_i \otimes w_j) \mid u \right\rangle}{M}$$

we obtain eq.(4).  $\square$

Thus the map

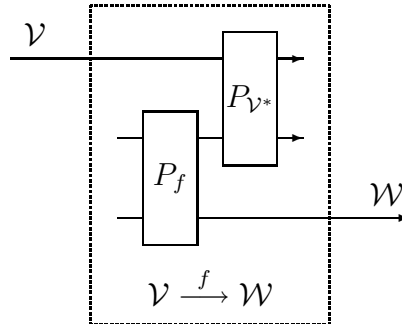
$$\xi \left( - \otimes \left( \sum_{i,j} f_{ij} \cdot (\bar{v}_i \otimes w_j) \right) \right) : \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{V}^* \otimes \mathcal{W}$$

produces  $f(-)$  as the last component of a pure tensor. The map

$$\zeta(- \otimes u) : \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{V}^* \otimes \mathcal{W}$$

does the same whenever  $u \not\perp \sum_{i,j} f_{ij} \cdot (\bar{v}_i \otimes w_j)$  (that is  $K \neq 0$ ). The function  $f$  is again encoded up to rescaling and a phase factor.

We can now represent the obtained result graphically as



with the contexts starting and ending within the dotted lines.

### 3 Physical realization of FDVec and FRel

We present an elementary version of quantum theory, restricted to finite-dimensional Hilbert spaces. For a standard textbook we refer to [16] and for more foundationally oriented texts to [19,20,24]. General quantum theory is formulated in terms of infinite-dimensional Hilbert spaces, but for many purposes in terms of insights, particular quantum features and even structural results, the infinite-dimensional aspects don't come in, e.g. Gleason's theorem [14] and the Kochen-Specker theorem [18], and the finite-dimensional restriction suffices for the physical realization of the traced monoidal category of finite-dimensional vector spaces.

#### 3.1 Elementary (superselection free) finite-dimensional quantum theory

Let  $\mathcal{H}$  be a finite-dimensional (complex) Hilbert space. Elements of  $\mathcal{H}$  are in the context of quantum theory frequently denoted by  $\psi$  and  $\phi$ . The description of a quantum system constitutes:

- (i) Description of the states of the system = kinematics.
- (ii) The description of evolution = reversible dynamics.
- (iii) The description of measurements = non-classical irreversible content.

The states of a quantum system encode as the set  $\Sigma_{\mathcal{H}}$  of one-dimensional subspaces of a Hilbert space  $\mathcal{H}$ , structured as an intersection system  $(\Sigma_{\mathcal{H}}, \mathcal{L}(\mathcal{H}))$  with  $\mathcal{L}(\mathcal{H})$  the orthocomplemented lattice of closed subspaces of  $\mathcal{H}$  — the classical counterpart to this is the intersection system  $(\Sigma, \mathcal{P}(\Sigma))$  with  $\mathcal{P}(\Sigma)$  the powerset of the states, that is a complete atomistic Boolean algebra. We motivate this below. Evolution between time  $t_1$  and  $t_2$  is described by a unitary operator, that is, a linear isomorphism that preserves the inner-product *i.e.* an automorphism of  $(\Sigma_{\mathcal{H}}, \mathcal{L}(\mathcal{H}))$ .

Measurements are represented by self-adjoint operators  $H : \mathcal{H} \rightarrow \mathcal{H}$ . When performing a measurement on the system in state  $\psi$  (a notational abuse that we from now on will use freely) where the corresponding self-adjoint operator has  $\sigma(H)$  as its spectrum of eigenvalues then we obtain as outcome of the measurement a value  $a \in \sigma(H)$  with corresponding probability  $Prob_{\psi}^H(a) = \langle \psi | P_a(\psi) \rangle = |P_a(\psi)|^2$ , where  $\psi$  is normalised and  $P_a$  is the projector on the subspace of eigenvectors with eigenvalue  $a$ . Note that  $\sum_{a \in \sigma(H)} P_a(\psi) = \psi$  and thus  $\sum_{a \in \sigma(H)} Prob_{\psi}^H(a) = 1$  since all eigenspaces  $V_a = \{\psi \in \mathcal{H} \mid P_a(\psi) = \psi\}$  are mutually orthogonal and span  $\mathcal{H}$ .

Sequential measurements obey von Neumann's projection postulate [19], that is, if the measurement yields  $a \in \sigma(H)$  as outcome then the state of the system changes from its initial state  $\psi$  to  $P_a(\psi)$ , so an immediate next measurement gives again  $a$  as outcome — since  $P_a(\psi)$  is itself an eigenvector with eigenvalue  $a$ . Thus, projectors encode true state transitions, explicitly

$$\tilde{P}_a : \Sigma_{\mathcal{H}} \setminus \mathcal{K} \rightarrow \Sigma_{\mathcal{H}} : \psi \mapsto P_a(\psi),$$

where

$$\mathcal{K} = \{\text{ray}(\psi) \in \Sigma_{\mathcal{H}} \mid \text{Prob}_{\psi}^H(a) = 0\}.$$

Note that any projector encodes itself a  $\{0, 1\}$ -valued measurement with  $V_0 \perp V_1$ . Moreover, they can be envisioned as encoding the primitive propositions on the systems since any self-adjoint operator  $H$  on a finite-dimensional Hilbert space can always be written as  $H = \sum_{a \in \sigma(H)} a P_a$  — this is the final dimensional version of von Neumann’s spectral decomposition theorem [19].<sup>5</sup> There is however more. The primitive propositions for a classical system are the subsets of the state space  $\Sigma$ . Indeed, let  $f : \Sigma \rightarrow \mathbb{R}$  be an observable of a classical system, then  $f^{-1}[E] \in \mathcal{P}(\Sigma)$  expresses for a state the property

“the value of function  $f$  is in  $E \subseteq \sigma(f)$ ”.

In the quantum case all statements of the form

“the value of self-adjoint operator  $H$  is in  $E \subseteq \sigma(H)$ ”

can be represented by the projector  $P_E^H = \sum_{a \in E} P_a^H$ , that has  $\bigoplus_{a \in E} V_a$  as fixpoints. Therefore, the subspaces encode the physical properties attributable to a quantum system. The projectors can then be envisioned as “verifications” [7, 20], or “preparations”, or “*active* specifications” in CS terms.

**Example:** *Polarization of photons.* Let  $Z$  be the axis of propagation. Consider as projector a light analyser that allows only vertically polarized light to pass, say polarized along the  $X$ -axis. If the in-coming light is polarized along the  $X$ -axis it passes (outcome 1). If the in-coming light is polarized along the  $Y$ -axis nothing passes (outcome 0). If it is polarized along an axis that makes an angle  $\theta$  with the  $X$ -axis then some light will pass, with relative amplitude  $\cos^2 \theta$ , and the light that passed will be vertically polarized. The amplitude reflects the quantum probability to pass, that is to obtain an outcome 1. The change of polarization angle from  $\theta$  to 0 is then the transition according to the projection postulate. The light analyser as such specifies that light is to be vertically polarized and does this in an active way.

A quantum system consisting of two subsystems is described in the tensor product  $\mathcal{H}_1 \otimes \mathcal{H}_2$  where  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are the Hilbert spaces in which we describe the respective subsystems. Thus, whereas in classical physics two systems are described by pairing states — the Cartesian product — in quantum theory we also have to consider superpositions of such pairs. Examples of projectors on  $\mathcal{H}_1 \otimes \mathcal{H}_2$  are those of the form  $P_1 \otimes P_2$ , explicitly definable as

$$(P_1 \otimes P_2)(\psi \otimes \phi) = P_1(\psi) \otimes P_2(\phi).$$

<sup>5</sup> Infinite-dimensionally, although self-adjoint operators (possibly only partially defined) might have no eigenstates at all, there always exists a projection valued measure on its spectrum, say  $P^H : \mathcal{B}(\sigma(H)) \rightarrow \mathbb{P}(\mathcal{H})$ , where  $\mathcal{B}(\sigma(H))$  denotes the Borel sets in  $\sigma(H)$  and  $\mathbb{P}(\mathcal{H})$  the projectors on  $\mathcal{H}$ , that reproduces the self-adjoint operator as  $\int_{\sigma(H)} a dP_a^H$ .

Even though values for general self-adjoint operators of this form should be envisioned as pairs  $(a_1, a_2) \in \sigma(H_1) \times \sigma(H_2)$  with corresponding probabilities

$$Prob_{\Psi}^{H_1 \otimes H_2}(a_1, a_2) = \langle \Psi | P_{a_1} \otimes P_{a_2} \Psi \rangle,$$

projectors compose conjunctively under  $\otimes$  that is  $(0, 1) \sim (0, 0) \sim (1, 0) \sim 0$  and  $(1, 1) \sim 1$ . Other examples of projectors on  $\mathcal{H}_1 \otimes \mathcal{H}_2$  are  $P_{\Psi} = \langle \Psi | - \rangle \Psi$  where  $\Psi$  cannot be written as a pure tensor. The examples in the previous section were all of this so-called “non-local” form.<sup>6</sup>

### 3.2 Physical realization of $(\mathbf{FDVec}, \otimes, Tr)$ .

The physical realization of  $\mathbf{FDVec}$  as a traced monoidal category consists of interpreting its objects, morphisms and additional operations, the tensor bifunctor and the trace, in physical terms, analogous to the realization of  $(\mathbf{Rel}, +)$  presented in the introduction in terms of tokens carrying data traveling in a network.

**i. “FDVec-Objects”.**  $\mathcal{V} \in Ob(\mathbf{FDVec})$  is interpreted/realized as a quantum system described in corresponding Hilbert space. We can represent this quantum system by its trajectory

$$\xrightarrow{\psi \in \mathcal{W}}$$

where the direction of the arrow should be read as the flow of time.

**ii. “FDVec-Morphisms”.**  $f \in \mathbf{FDVec}(\mathcal{V}, \mathcal{W})$  is interpreted as the process obtained by interpreting in Lemma 2 the vector spaces as quantum systems and the projectors as  $\{0, 1\}$ -measurements.<sup>7</sup>

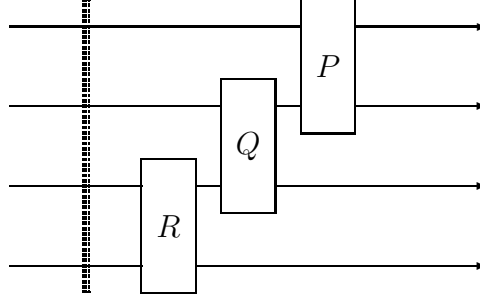
<sup>6</sup> ‘Non-local’ should not, in this context, be envisioned in space-like terms. Non-local unitary operations are considered in quantum control theory, quantum computation and quantum information [6], when the system evolves according to a non-local Hamiltonian, i.e.  $U(t) = e^{\frac{i}{\hbar} H t} \neq U_1(t) \otimes U_2(t) = e^{\frac{i}{\hbar} H_1 t} \otimes e^{\frac{i}{\hbar} H_2 t}$ . They allow to obtain non-local projectors as  $U.(P_1 \otimes P_2).U^{-1}$  when  $U \neq U_1 \otimes U_2$ . Any one-dimensional projector  $P$  on  $\mathcal{H}_1 \otimes \mathcal{H}_2$  can be obtained in that way as  $U.P_{\psi \otimes \phi}.U^{-1}$  taking  $U$  such that  $U^{-1}(\psi \otimes \phi)$  is a fixpoint of  $P$  since  $P_{\psi \otimes \phi} = P_{\psi} \otimes P_{\phi}$ . Note that one cannot obtain arbitrary projectors in this way due to the simple fact that the dimension of the projector on the global space should factor in a product of the dimensions of the underlying ones, e.g a projector on a 5-dimensional subspace (with 7-dimensional orthocomplement) in case of  $\dim(\mathcal{H}_1) = 3$  and  $\dim(\mathcal{H}_2) = 4$ .

<sup>7</sup> Since states encode in Hilbert space as rays rather than as vectors, two functions  $f$  and  $g$  which are such that  $f = r e^{i\theta} \cdot g$  with  $r \in \mathbb{R}_+$  and  $\theta \in [0, 2\pi[$  will be encoded by the same physical process. Due to linearity of the maps however, coincidence of the action of  $f$  and  $g$  on one-dimensional subspaces does force them to be essentially the same [10,11,12]. Quantum mechanics moreover provides sufficient tools to even encode phase factors in a measurable way by de-localising one part of the state from another part [21]. Relative amplitudes can be envisioned as relative densities. But, more importantly, the minor unfaithfulness, when retained, is not essential at all for the qualitative results we pursue and does not seem to play any role in any applications we can think of.

iii. **“FDVec-Tensor”**.  $\mathcal{V} \otimes \mathcal{W} \in \text{Ob}(\mathbf{FDVec})$  is interpreted as a compound quantum system which extends to  $f \otimes g \in \mathbf{FDVec}(\mathcal{V} \otimes \mathcal{V}', \mathcal{W} \otimes \mathcal{W}')$  as non-interacting parallel composition. The network consists of a number of parallel tracks on which quantum particles synchronously travel “as a wave”

$$\sum_{i_1, \dots, i_n} c_{i_1 \dots i_n} \psi_{i_1} \otimes \dots \otimes \psi_{i_n},$$

being acted on “locally” (that is on part of the wave) by processes.



The dashed line represents a propagating wave front.

A succesful pass of a wave of particles through the network requires a succesful pass through every projector. Note that the propagation of the wave can as well be envisioned as being active and passive, that is, either the particles propagate themselves or the processes acting on them.

Note that in these physical terms Lemma 2 embodies what could be called “generalised probabilistic teleportation”, with “probabilistic quantum teleportation” and “probabilistic quantum cloning” as instances.<sup>8</sup> It suffices to set respectively

$$f = id_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{V} : \psi_i \mapsto \psi_i$$

and

$$f = \Delta_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{V} : \psi_i \mapsto (\psi_i, \psi_i)$$

with  $\{\psi_i\}_i$  a base of  $\mathcal{V}$ .

iv. **“FDVec-Trace”**.  $Tr_{\mathcal{V}, \mathcal{W}}^{\mathcal{U}}$  for  $f \in \mathbf{FDVec}(\mathcal{V} \otimes \mathcal{U}, \mathcal{W} \otimes \mathcal{U})$  is interpreted as the process obtained from Lemma 1 by interpreting the vector spaces as quantum systems and the projectors as measurements, with again the same remark on the weights  $N$  as above.<sup>9</sup>

<sup>8</sup> The no-cloning theorem for quantum states prohibits perfect (certain) copying of quantum states by means of ‘physical operations’, usually envisioned as trace preserving completely positive maps on density matrices — see for example [25] or [13]. For the standard quantum teleportation protocol we refer to [5].

<sup>9</sup> Note here the fundamental difference between the vector space trace with respect to this realisation and the realization of the partial trace for unitary operations acting on so-called bipartite states (as encountered in standard quantum information textbooks, e.g. for the realization of superoperators). Here we sum over pure states where the usual partial trace realization yields a sum over density operators. In general, that partial trace realization also does not de-entangle.



Since we realized all operations that enable the GoI-construction, we realized the GoI-construction for **FDVec** itself. For the whole interpretation we need only state update (projectors) and no unitary operators, which are themselves morphisms realized in terms of state update.

Note in particular that all processes in the network can be reduced (via Lemma 2) to the form  $P_{\mathcal{U}^*}$  or  $P_f$ , where we even have  $P_{\mathcal{U}^*} = P_{id_{\mathcal{U}^*}}$ . In full notation we as such only used

$$P_f : \mathcal{V} \rightarrow \mathcal{W} \quad \text{and} \quad P_{id} : \mathcal{U} \rightarrow \mathcal{U}.$$

Thus, all processes can be typed by the Linear Logic encoded in **FDVec**.

Note here also that the isomorphism  $[\mathcal{V}, \mathcal{W}] \cong \mathcal{V}^* \otimes \mathcal{W}$  is physically exploited as an isomorphism between application of arbitrary linear functions and projection on corresponding entangled states. This suggests the slogan that “basic types of quantum theory are themselves actions”, represented by projections. The projectors of the form  $P_f$  then encode specification of a particular type of entanglement, where the corresponding action itself is then the process of entangling.

### 3.3 Physical realization of $(\mathbf{FRel}, \times, Tr)$

We will now establish a functorial mapping that carries the trace structure of  $(\mathbf{FDVec}, \otimes)$  on  $(\mathbf{FRel}, \times)$  and hence also the physical realization. We recall that **FRel** is equivalently described as the Kleisli category of the covariant powerset monad on **FSet**, or equivalently again the category of finitely-generated free suplattices and their homomorphisms. Concretely, the isomorphism from **FRel** to this latter category is described as follows. Sets  $X$  in **FRel** are represented as powersets  $\mathcal{P}(X)$  in  $\mathbf{P}(\mathbf{FSet})$  and relations  $R \subseteq X \times Y$  as maps

$$f_R : \mathcal{P}(X) \rightarrow \mathcal{P}(Y) : T \mapsto \{y \in Y \mid \exists t \in T : (t, y) \in R\}.$$

Assume that every vector space  $\mathcal{V}$  has a specified base  $\{e_i\}_{i \in X}$ , explicitly denoted as  $(\mathcal{V}, \{e_i\}_{i \in X})$ , and that for  $(\mathcal{V}, \{e_i\}_{i \in X})$  and  $(\mathcal{W}, \{e'_j\}_{j \in Y})$  the base of  $\mathcal{V} \otimes \mathcal{W}$  is  $\{e_i \otimes e'_j\}_{(i,j) \in X \times Y}$ . Define (with abuse of the notation **FDVec**):

$$\tilde{R} : \mathbf{FDVec} \rightarrow \mathbf{FRel} : \begin{cases} (\mathcal{V}, \{e_i\}_{i \in X}) \mapsto X \\ f \mapsto \tilde{R}_f := \{(i, j) \in X \times Y \mid f_{ij} \neq 0\}. \end{cases}$$

Since

$$\begin{aligned} \tilde{R}_{f,g} &= \{(i, k) \in X \times Z \mid \sum_j f_{ij} g_{jk} \neq 0\} \\ &\subseteq \{(i, k) \in X \times Z \mid \exists j \in Y : f_{ij} g_{jk} \neq 0\} = \tilde{R}_f; \tilde{R}_g \end{aligned}$$

and

$$\tilde{R}_{id_X} = \{(i, j) \in X \times X \mid \delta_{ij} \neq 0\} = \Delta_X$$

it follows that  $\tilde{R}$  is a lax functor. To show that it is not a (strict) functor, it suffices to consider  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with  $f_{ij} = \frac{1}{\sqrt{2}}$  except for  $f_{22} = -\frac{1}{\sqrt{2}}$ . Then we have  $f;f = id_{\mathbb{R}^2}$  so clearly  $R_{f;f} \neq R_f;R_f$ . So the presence of negative values, or more general, complex phases, allows “mutual cancelation”, a qualitative feature of **FDVec** that has no counterpart in **FRel**. When restricting to those morphisms in **FDVec** which only have non-negative real coefficients  $f_{ij}$  we have  $\sum_j f_{ij}g_{jk} \neq 0 \Leftrightarrow \exists j \in Y : f_{ij}g_{jk} \neq 0$  and thus obtain a true functor. Denote this category by **FDVec**<sub>+</sub> and from now on we set  $R : \mathbf{FDVec}_+ \rightarrow \mathbf{FRel}$  for the restriction. This functor preserves the tensor by construction and also preserves the trace. Indeed,

$$\begin{aligned} R_{Tr_{V,W}^U f} &= \{(i, j) \in X \times Y \mid \sum_k f_{ikjk} \neq 0\} \\ &= \{(i, j) \in X \times Y \mid \exists k \in Z : f_{ikjk} \neq 0\} = Tr_{RV, RW}^{RU} R_f \end{aligned}$$

since

$$R_f = \{((i, l), (j, k)) \in (X \times Z) \times (Y \times Z) \mid f_{iljk} \neq 0\}.$$

Concretely, in terms of  $P(\mathbf{FSet})$  rather than **FRel**, the above functor assigns to  $v \in (V, \{e_i\}_{i \in X})$  the set of all  $i \in X$  such that  $\langle v | e_i \rangle \neq 0$ , idem ditto for the images of the base vectors under linear maps, so in particular we have

$$(R_f)(i) = \{j \in Y \mid f_{ij} \neq 0\}$$

and in general for  $T \subseteq X$

$$(R_f)(T) = \{j \in Y \mid \exists i \in T : f_{ij} \neq 0\}.$$

*This concreteness is essential.* It enables us to define a system with states given by equivalence classes of states of a quantum system, which we will denote by elements of  $\mathcal{P}(X)$ : The system is in state  $T \in \mathcal{P}(X)$  if the quantum system is in a state of the form  $\sum_{i \in T} c_i \psi_i$  with all  $c_i$  non-zero. The physically meaningful operations for the quantum system, in vector space terms composition, tensor and trace, are all preserved. As such we realize  $(\mathbf{FRel}, \times, Tr)$  physically.

Note that we could not obtain a realization via embedding, since there exists no functorial embedding  $F : P(\mathbf{FSet}) \rightarrow \mathbf{FDVec}$  that assigns to each  $\{i\} \in \mathcal{P}(X)$  a corresponding base vector  $e_i \in \{e_j\}_j$ . This fact reflects a “resource sensitivity” of linear maps, that has no counterpart for relations, in the sense of “how many elements in the argument contribute to a particular element of the image”. This is exactly what is captured by so-called multirelations, that is, the category **Mult** <sub>$\mathbb{N}$</sub>  of sets with as morphisms maps  $r : X \times Y \rightarrow \mathbb{N} : (i, j) \mapsto r_{ij}$ . In the above setting, this category can be made concrete when restricting to finite sets by envisioning the sets  $X$  themselves as the collection of maps

$$\mathcal{P}_{\mathbb{N}}(X) := \{t : X \rightarrow \mathbb{N} : i \mapsto t_i\}$$

on which the multirelations act as

$$f_r : \mathcal{P}_{\mathbb{N}}(X) \rightarrow \mathcal{P}_{\mathbb{N}}(Y) : t \mapsto f_r(t) := Y \rightarrow \mathbb{N} : j \mapsto \sum_{i \in X} t_i r_{ij}$$

Note that the same construction still holds when considering  $\mathbb{R}_+$  instead of  $\mathbb{N}$ . Denoting continuously valued multirelations on finite sets as  $\mathbf{FMult}_{\mathbb{R}_+}$  we obtain  $\mathbf{FDVec}_+ \cong \mathbf{FMult}_{\mathbb{R}_+}$  which itself is then naturally equipped with tensor and trace.

It is possible to choose the bases  $\{e_i\}_{i \in X}$  such that  $\mathbf{FDVec}_+$  is closed under the canonical  $\mathbf{FDVec}$   $*$ -operation. It suffices to set  $(\mathcal{V}^*, \{\bar{e}_i\}_{i \in X})$  whenever we have  $(\mathcal{V}, \{e_i\}_{i \in X})$ . Then we have that  $f_{ij}^* = f_{ji}$ , as such inheriting being positive reals and moreover, the induced  $\mathbf{FDVec}_+$   $*$ -operation is preserved by the functor  $R$  since the canonical  $*$ -operation in  $(\mathbf{FRel}, \times)$  on objects is the identity and on morphisms is the relational converse.

### 3.4 Delineation of qualitative computational differences

The above construction enables to delineate qualitative differences between quantum and classical process networks, where we envision the latter described by  $(\mathbf{Rel}, +)$  as discussed in the introduction. We will proceed via a two step “descent”  $(\mathbf{FDVec}, \otimes) \mapsto (\mathbf{Rel}, \times) \mapsto (\mathbf{Rel}, +)$  in terms of their physical realizations. Following section 3.3, the passage from  $(\mathbf{FDVec}, \otimes)$  to  $(\mathbf{Rel}, \times)$  via the functor  $R$  goes with the following “loss”:

- Specification of phase-factors due to the domain restriction of  $R$  to  $\mathbf{FDVec}_+$ , corresponding to the loss of the possibility of mutual cancellation. In view of the importance of “mutual cancellation” in known quantum computational algorithms this feature is essential. Note that this distinction also appears when comparing probabilistic and quantum Turing machines.
- Expressibility of multiplicities since  $R$  is not faithful, corresponding to the loss of degrees of freedom at the level of the system traveling through the network.
- Processing along multiple incompatible bases becomes impossible, corresponding to the loss of degrees of freedom at the information processing level.

Next we discuss the passage from  $(\mathbf{Rel}, \times)$  and  $(\mathbf{Rel}, +)$  where we encounter two crucial physical differences.

- In the case of the realization of  $(\mathbf{Rel}, +)$  multiple images for an initial state of a process stands for “non-determinism”, in the case of the realization of  $(\mathbf{Rel}, \times)$  it stands for “doing multiple things together” via creation of a superposition state that reflects all possible images for a single argument.
- In the case of the realization of  $(\mathbf{Rel}, +)$  the trace is realized recursively, in the case of the realization of  $(\mathbf{Rel}, \times)$  it is a one-shot passage, with corresponding implications in terms of complexity.

## 4 Applications

The present paper, while very much a first step, lays the basis for a number of further developments. We realized wave-style GoI models of Multiplicative Linear Logic via the quantum physical processes *entangling* and *de-entangling* by means of typed projectors. In particular, we can now give a ‘physical realization’ of proofs in the Multiplicative fragment of Linear Logic, or of terms in the (simply-typed) *affine*  $\lambda$ -calculus, as quantum systems. Since e.g. boolean circuits can easily be represented as affine lambda terms, this gives us a ‘compilation process’ taking high-level functional programs into quantum systems, of a form which looks very different to the current low-level descriptions of quantum algorithms and machine models. It will surely be of interest to look at examples of this compilation process and compare them to current approaches. Since there are currently very few quantum algorithms, it is to be hoped that such higher-level methods will be fruitful in suggesting new ideas and approaches. We can also pose the following questions for future investigation.

- Can we realize proof reduction (normalization) in linear  $\lambda$ -calculus in constant time via physical processes, due to the one shot trace of the computational model proposed in this paper?
- Can one produce a general picture of wave-style GoI in terms of quantum structures? A possible candidate is a category of orthoalgebras, which captures both sets and general quantum structures, and admits a tensor product. See for example [9].

## References

- [1] Abramsky, S., *Retracing some Paths in Process Algebra*, “Proceedings of the Seventh International Conference on Concurrency Theory” LNCS **1119** (1996), 1–17.
- [2] Abramsky, S., E. Haghverdi and P. J. Scott, *Geometry of Interaction and Linear Combinatory Algebras*, Mathematical Structures in Computer Science, To appear.
- [3] Abramsky, S. and R. Jagadeesan, *New Foundations for the Geometry of Interaction*, Information and Computation **111** (1994), 53–119.
- [4] Bainbridge, E. S., *Feedback and Generalized Logic*, Information and Control **31** (1976), 75–96.
- [5] Bennet, C. H., C. Brassard, C. Crépeau, R. Jozsa, A. Peres and W. K. Wothers, *Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels*, Physical Review Letters **70** (1993), 1895–1899.

- [6] Bennett, C. H., J. I. Cirac, D. Leifer, D. W. Leung, N. Linden, S. Popescu and G. Vidal, *Optimal Simulation of Two-Qubit Hamiltonians using General Local Operations*, URL: <http://arXiv.org/abs/quant-ph/0107035>.
- [7] Birkhoff, G. and J. von Neumann, *The Logic of Quantum Mechanics*, Annals of Mathematics **37** (1936), 823–843.
- [8] Blute, R. F., I. T. Ivanov and P. Panangaden, *Discrete Quantum Causal Dynamics*, URL: <http://arXiv.org/abs/quant-ph/0109053>.
- [9] Coecke, B., D. J. Moore and A. Wilce, Eds., “Current Research in Operational Quantum Logic: Algebras, Categories, Languages”, Kluwer Academic Publishers, 2000.
- [10] Faure, Cl.-A. and A. Frölicher, *Morphisms of Projective Geometries and of Corresponding Lattices*, Geometriae Dedicata **47**, (1993) 25–40.
- [11] Faure, Cl.-A. and A. Frölicher, *Morphisms of Projective Geometries and Semilinear Maps*, Geometriae Dedicata **53**, (1994) 237–262.
- [12] Faure, Cl.-A. and A. Frölicher, “Modern Projective Geometry”, Kluwer Academic Publishers, 2000.
- [13] Jozsa, R., *A Stronger No-Cloning Theorem*, URL: <http://arXiv.org/abs/quant-ph/0204153>.
- [14] Gleason, A. M., *Measures on the Closed Subspaces of a Hilbert Space*, Journal of Mathematics and Mechanics **6** (1957), 885–893.
- [15] Haghverdi, E., “A Categorical Approach to Linear Logic, Geometry of Proofs and Full Completeness”, PhD Thesis, University of Ottawa, 2000.
- [16] Isham, C. J., “Lectures on Quantum Theory”, Imperial College Press, 1995.
- [17] Joyal, A., R. Street and D. Verity, *Traced Monoidal Categories*, Proceedings of the Cambridge Philosophical Society **119** (1996), 447–468.
- [18] Kochen, S. and E. P. Specker, *The Problem of Hidden Variables in Quantum Mechanics*, Journal of Mathematics and Mechanics **17**, (1967) 59–87.
- [19] von Neumann, J., “Mathematische Grundlagen der Quantenmechanik”, Springer-Verlag, 1932. Translation in “Mathematical Foundations of Quantum Mechanics”, Princeton University Press, 1955.
- [20] Piron, C., “Foundations of Quantum Physics”, W.A. Benjamin, 1976.
- [21] Rauch, H., W. Treimer and U. Bonse, *Test of a Single Crystal Neutron Interferometer*, Physics Letters **47 A**, (1974) 369–384.
- [22] Selinger, P., *A Note on Bainbridge’s Powerset Construction*, URL: <ftp://quasar.mathstat.uottawa.ca/pub/selinger/bainbridge.ps.gz>.
- [23] Selinger, P., *Categorical Structure of Asynchrony*, “Proceedings of MFPS 15”, Electronic Notes in Theoretical Computer Science **20** (1999).

- [24] Varadarajan, V. S., “The Geometry of Quantum Theory”, Springer-Verlag, 1968.
- [25] Wootters, W. and W. Zurek, *A Single Quantum cannot be Cloned*, Nature **299** (1982), 802–803.